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Conversion between two bases of rotationally symmetric spheroidal vector wavefunctions

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Abstract

A conversion between two eigenfunction bases of spheroidal vector wavefunctions is described, both bases comprising two independent sets of solutions for the electro-magnetic vector wave equation with azimuthal symmetry. A set of definite integrals over the angular variable, arising in the conversion is evaluated in closed form, each integral containing a product of two angular spheroidal wavefunctions with $m = 0$ or 1, weighted by algebraic functions.

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1. Introduction

The electromagnetic inverse problem of reconstructing time-harmonic minimum energy current distributions from the scattered field by means of linear inversion theory requires the computation of certain inner products of a pair of vector eigenfunctions. Recently, in the course of extending a source reconstruction procedure for spherical scatterers [1] to spheroidal ones, the author came across the problem of computing such inner products for spheroidal vector eigenfunctions.

Let $\mathbf{X}(\mathbf{r})$ and $\mathbf{Y}(\mathbf{r})$ denote two solenoidal vector wavefunctions, satisfying the time-harmonic Helmholtz wave equation for the electric field

$$(\nabla \times \nabla \times \overline{\overline{I}} - k^2\overline{\overline{I}}) \cdot \mathbf{E} = 0 \tag{1}$$

in the spheroidal system of co-ordinates $(f; \xi, \eta, \varphi)$, where ξ and η are the radial and angular spheroidal co-ordinates, φ being the azimuth angle and f is the semifocal distance [2, 3]. Then the inner products to be evaluated are the volume integrals

$$\langle \mathbf{X}|\mathbf{Y}\rangle_V = f^3 \int_0^{2\pi} \int_{-1}^{1} \int^{\xi_0} \mathbf{X}^\star \cdot \mathbf{Y}(\xi^2 \mp \eta^2)\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\varphi \tag{2}$$

the lower endpoint of the ξ-integral being 1 for prolate (elongated) and 0 for oblate (flattened) spheroids (the upper sign pertains to prolate and the lower to oblate spheroids). The spheroid, presumed to be aligned with the z-axis, is completely characterized by the interfocal distance $2f$ and the co-ordinate of the surface: $\xi_0 = a/f$. For prolates, $f = \sqrt{a^2 - b^2}$, and for oblates $\sqrt{b^2 - a^2}$, a and b representing the polar and equatorial radius.

In the spheroidal system of co-ordinates, several eigenfunctions bases are available [2, 3], some being computationally more advantageous than others. Eigenfunctions for the vector field problems of electromagnetic radiation [1], scattering [3, 4] and cavity resonance [5] are usually derived for the general wave equation (1), while those specifically chosen for the analyses of omnidirectional spheroidal antennas, e.g. [6], are typically developed to satisfy the rotationally symmetric (φ-independent) version of the same directly. One of the peculiarities of the spheroidal system of co-ordinates is that the base of vector wavefunctions obtained by the former method is not identical, in fact not even closely related, to that obtained from the latter, albeit including the same vector components and behaving spatially in a similar way. The latter eigenfunctions are not only less complicated than the former, but also do they not contain any potential singularities that would endanger an evaluation of the volume integrals of the inner products as independent, one-dimensional integrals. Hence, a conversion between two alternative bases of eigenfunctions for rotationally symmetric spheroidal waves would be valuable. Such a conversion is put forward in this paper, and the relevant integrals for the angular spheroidal wavefunctions are evaluated in closed form.

2. Eigenfunction bases

Unlike the spherical wavefunctions, for which the volume integrals corresponding to (2) render zero except when $\mathbf{X} = \mathbf{Y}$ (meaning, *ipso facto*, that every spherical vector wavefunction is perfectly orthogonal to all the others), any two representatives for the families of divergenceless vector eigenfunctions $\{\mathbf{M}_{mn}, \mathbf{N}_{mn}\}$ possible to construct within the spheroidal system in the form

$$\mathbf{M}_{mn}(kf; \xi, \eta, \varphi) = \nabla \times (\Psi_{mn}\hat{\mathbf{a}}) \tag{3}$$

$$\mathbf{N}_{mn}(kf; \xi, \eta, \varphi) = \frac{1}{k}\nabla \times \nabla \times (\Psi_{mn}\hat{\mathbf{a}}) = \frac{\nabla \times \mathbf{M}_{mn}}{k} \tag{4}$$

are generally neither orthogonal to each other, nor to the other elements of the set [2, 3]. The eigenfunctions are, however, clearly transverse to each other, $\mathbf{M}_{mn}$ being furthermore transverse to $\hat{\mathbf{a}}$. Thus, assuming that the eigenfunctions represent the electric field, the $\mathbf{M}_{mn}$ modes are labelled transverse electric (with respect to $\hat{\mathbf{a}}$) and the $\mathbf{N}_{mn}$ modes, correspondingly, transverse magnetic.

The function Ψ_{mn} in (3) and (4)

$$\Psi_{mn}(kf; \xi, \eta, \varphi) = R_{mn}^{(1)}(kf; \xi) S_{mn}(kf; \eta) \exp(-jm\varphi)$$

is a solution of the scalar wave equation, with S_{mn} and $R_{mn}^{(1)}$ denoting, respectively, the angular and radial spheroidal wavefunctions. To ensure that the field inside the spheroid is regular everywhere the radial functions must be of the first kind. The vector eigenfunctions (3) and (4) are thus applicable to determining resonant modes of a spheroidal cavity [5]. Outside the spheroid, a complete set of eigenfunctions would also have to involve radial functions of the second kind (which are infinite at the origin), but these are not treated here.

The spheroidal wavefunctions, S_{mn} and $R_{mn}^{(1)}$, are defined in terms of Legendre functions of the first kind, P_n^m, and spherical Bessel functions, $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x)$, ($J_n$ being the conventional cylindrical Bessel function) as

$$S_{mn}(kf;\eta) = \sum_{r=0}^{\infty} d_r^{|m|n}(kf) P_{|m|+r}^{|m|}(\eta) \tag{5}$$

$$R_{mn}^{(1)}(kf;\xi) = \frac{(n-|m|)!}{(n+|m|)!} \left(\frac{\xi^2-1}{\xi^2}\right)^{\frac{|m|}{2}} \sum_{r=0}^{\infty} j^{r+|m|-n} \frac{(2|m|+r)!}{r!} d_r^{|m|n}(kf) j_{|m|+r}(kf\xi). \tag{6}$$

The expansion coefficients $d_r^{|m|n}(kf)$ (the parameter kf is henceforth implicit) common for (5) and (6) are obtained recursively using the eigenvalues of the governing differential equation [2, 7]. To ensure a sufficient accuracy of the angular integrals to be treated in sections 2.1 and 3.1, special care must be taken to determine the coefficients $d_r^{|m|n}$ as precisely as possible (for instance, by suitably combining forward and backward recursions).

Regardless the choice for the vector $\hat{\mathbf{a}}$—be it $\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z$ or $\mathbf{u}_\xi, \mathbf{u}_\eta, \mathbf{u}_\varphi$ or, as in our case, the position vector $\mathbf{r}$—numerical evaluation of three-dimensional integrals cannot be avoided. In this respect, the rotationally symmetric modes, lacking φ-dependence and being thus characterized by the azimuthal index $m = 0$, form a theoretically interesting but nonetheless practically important exception. The two sets of spheroidal vector eigenfunctions (labelled by the superscript A) of the base of rotationally symmetric modes are

$$\mathbf{M}_{0n}^A = \nabla \times (\Psi_{0n}\mathbf{r}) = \frac{\sqrt{(1-\eta^2)(\xi^2 \mp 1)}}{\xi^2 \mp \eta^2} \left[\eta S_{0n}(\eta) \frac{\partial R_{0n}^{(1)}(\xi)}{\partial \xi} - \xi R_{0n}^{(1)}(\xi) \frac{\partial S_{0n}(\eta)}{\partial \eta} \right] \mathbf{u}_\varphi \tag{7}$$

$$\begin{aligned} \mathbf{N}_{0n}^A = {} & \frac{\sqrt{\xi^2 \mp 1}}{kf\sqrt{\xi^2 \mp \eta^2}} \frac{\partial}{\partial \eta} \left[\frac{(1-\eta^2)}{(\xi^2 \mp \eta^2)} \left[\eta S_{0n}(\eta) \frac{\partial R_{0n}^{(1)}(\xi)}{\partial \xi} - \xi R_{0n}^{(1)}(\xi) \frac{\partial S_{0n}(\eta)}{\partial \eta} \right] \right] \mathbf{u}_\xi \\ & - \frac{\sqrt{1-\eta^2}}{kf\sqrt{\xi^2 \mp \eta^2}} \frac{\partial}{\partial \xi} \left[\frac{(\xi^2 \mp 1)}{(\xi^2 \mp \eta^2)} \left[\eta S_{0n}(\eta) \frac{\partial R_{0n}^{(1)}(\xi)}{\partial \xi} - \xi R_{0n}^{(1)}(\xi) \frac{\partial S_{0n}(\eta)}{\partial \eta} \right] \right] \mathbf{u}_\eta. \end{aligned} \tag{8}$$

Although these sets are orthogonal by direction, they are still not orthogonal to the other elements of the same set. Thus, the remaining nonzero inner products are $\langle \mathbf{M}_{0n}^A | \mathbf{M}_{0n'}^A \rangle$ and $\langle \mathbf{N}_{0n}^A | \mathbf{N}_{0n'}^A \rangle$. Unfortunately, the factor $\xi^2 \mp \eta^2$ appearing in the denominators of the respective double integral (the φ-integral can be readily integrated) precludes an independent evaluation of the integrals for the variables ξ and η. Moreover, the factor $\xi^2 - \eta^2$ (for prolates) tends to zero at $(\xi, \eta) = (1, \pm 1)$, as does the factor $\xi^2 + \eta^2$ (for oblates) at $(\xi, \eta) = (0, 0)$, which makes numerical evaluation of the integrals a delicate matter.

Fortunately, an alternative base of eigenfunctions can be employed, for which the inner product separates into two independent integrals, one for each variable. The alternative pair of solenoidal eigenfunctions, which can be derived directly from the φ-independent version of the wave equation (1) [8, 9], and being in that respect equally capable of expressing any rotationally symmetric and regular electromagnetic field inside a given spheroid as the pair (7) and (8), is [8, 9]

$$\mathbf{M}_{1n}^B = R_{1n}^{(1)}(\xi) S_{1n}(\eta) \mathbf{u}_\varphi \tag{9}$$

$$\mathbf{N}_{1n}^B = \frac{(kf)^{-1}}{\sqrt{\xi^2 \mp \eta^2}} \left[R_{1n}^{(1)}(\xi) \frac{\partial}{\partial \eta}\big(S_{1n}(\eta)\sqrt{1-\eta^2}\big) \mathbf{u}_\xi - S_{1n}(\eta) \frac{\partial}{\partial \xi}\big(R_{1n}^{(1)}(\xi)\sqrt{\xi^2 \mp 1}\big) \mathbf{u}_\eta \right]. \tag{10}$$

This base (denoted by the superscript B) is notably simpler in appearance, and also computationally more attractive, than the base A. The properties of the base B eigenfunctions have been examined by Wall [9] in the course of formulating an electromagnetic Green tensor in the spheroidal system, and their travelling wave alternatives (the function $R_{1n}^{(4)}$ replacing $R_{1n}^{(1)}$) have been widely applied in antenna theory [6]. However, the functions (9) and (10) of base B are in no way related to the functions (7) and (8) of the base A, since veritable recursion formulae linking spheroidal wavefunctions of different n or m indices do not exist [2, chapter 7]. Hence, a conversion routine would be of use for expressing the eigenfunctions A with those of the simpler eigenfunctions B. Such a routine will be presented in the following section.

For the eigenfunction pair B the mentioned inner products are easily written in terms of separate integrals. In the following, we show how each of the integrals over the angular variable η may be evaluated in closed form.

2.1. Angular integrals involved in $\langle \mathbf{M}_{1n}^B | \mathbf{M}_{1n'}^B \rangle$ *and* $\langle \mathbf{N}_{1n}^B | \mathbf{N}_{1n'}^B \rangle$

To evaluate

$$I_1(n, n') = \int_{-1}^{1} S_{1n}(\eta) S_{1n'}(\eta)\, \mathrm{d}\eta \tag{11}$$

we refer to the fact that orthogonality of the angular spheroidal wavefunctions, which is due to the corresponding differential equation being a special case of the Sturm-Liouville equation, assures that expression (11) is zero except if $n = n'$. When that is the case, equation (11) renders a constant Λ_n, which for fixed kf only depends on n, and which can be obtained using (5) and the orthogonality of Legendre functions [9] as

$$\Lambda_n = 2 \sum_{r=0}^{\infty} \left(d_r^{1n}\right)^2 \frac{(r+2)(r+1)}{(2r+3)} \tag{12}$$

so that

$$I_1(n, n') = \delta_{n,n'} \Lambda_n. \tag{13}$$

In the evaluation of

$$I_2(n, n') = \int_{-1}^{1} S_{1n}(\eta) S_{1n'}(\eta) \eta^2\, \mathrm{d}\eta = \sum_{r=0}^{\infty} d_r^{1n} \sum_{r'=0}^{\infty} d_{r'}^{1n'} \int_{-1}^{1} P_{r+1}^1(\eta) P_{r'+1}^1(\eta) \eta^2\, \mathrm{d}\eta \tag{14}$$

we are led to use the identity

$$\int_{-1}^{1} P_r^1(x) P_{r'}^1(x) x^2\, \mathrm{d}x = \begin{cases} \dfrac{2r(r+1)(2r^2+2r-3)}{(2r-1)(2r+1)(2r+3)} & r = r' \\ \dfrac{2r(r+1)(r+2)(r+3)}{(2r+1)(2r+3)(2r+5)} & r = r'-2 \\ \dfrac{2(r-2)(r-1)r(r+1)}{(2r-3)(2r-1)(2r+1)} & r = r'+2 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

which is derived in the appendix. As a result, equation (14) can be written as a single sum

$$I_2(n, n') = 2 \sum_{r=0}^{\infty} \frac{(r+1)(r+2)}{(2r+3)} d_r^{1n} \times \left[\frac{(2r^2+6r+1)}{(2r+1)(2r+5)} d_r^{1n'} + \frac{(r^2+7r+12)}{(2r+5)(2r+7)} d_{r+2}^{1n'} + \frac{(r-1)r}{(4r^2-1)} d_{r-2}^{1n'} \right]. \tag{16}$$

Lastly, the angular integral

$$I_3(n,n') = \int_{-1}^{1} \frac{\mathrm{d}}{\mathrm{d}\eta}\big[\sqrt{1-\eta^2}S_{1n}(\eta)\big]\frac{\mathrm{d}}{\mathrm{d}\eta}\big[\sqrt{1-\eta^2}S_{1n'}(\eta)\big]\,\mathrm{d}\eta$$
$$= \sum_{r=0}^{\infty} d_r^{1n} \sum_{r'=0}^{\infty} d_{r'}^{1n'} \int_{-1}^{1} \frac{\mathrm{d}}{\mathrm{d}\eta}\big[\sqrt{1-\eta^2}P_{r+1}^1(\eta)\big]\frac{\mathrm{d}}{\mathrm{d}\eta}\big[\sqrt{1-\eta^2}P_{r'+1}^1(\eta)\big]\,\mathrm{d}\eta \tag{17}$$

suggests the use of the orthogonality result

$$\int_{-1}^{1} \frac{\mathrm{d}}{\mathrm{d}x}\big[\sqrt{1-x^2}P_r^1(x)\big]\frac{\mathrm{d}}{\mathrm{d}x}\big[\sqrt{1-x^2}P_{r'}^1(x)\big]\,\mathrm{d}x = \frac{2(r+r^2)^2}{2r+1}\delta_{r,r'} \tag{18}$$

also derived in the appendix. In sum,

$$I_3(n,n') = 2\sum_{r=0}^{\infty} d_r^{1n} d_r^{1n'} \frac{(r+1)^2(r+2)^2}{2r+3}. \tag{19}$$

3. Conversion of eigenfunction base

If the conversion be of the form

$$\mathbf{M}_{0n}^A = \sum_{n'=1}^{N} \alpha_{nn'}\mathbf{M}_{1n'}^B \tag{20}$$

our task is to determine the conversion coefficients $\alpha_{nn'}$ so that (20) is valid for a given spheroid. The truncation mode index N may be chosen at will, but to guarantee the stability of the inverse solution [1] a practicable limit, which may be exceeded by a few modes only, is the 'electrical size' of the spheroid; $N = kf\xi$.

Taking the inner product of (20) and $\mathbf{M}_{1n'}^B$, for $n, n' = 1, 2, \ldots, N$, with respect to this spheroid we have

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1N} \\ \vdots & \ddots & \vdots \\ \alpha_{N1} & \cdots & \alpha_{NN} \end{pmatrix} = \begin{pmatrix} \langle \mathbf{M}_{01}^A | \mathbf{M}_{11}^B \rangle & \cdots & \langle \mathbf{M}_{01}^A | \mathbf{M}_{1N}^B \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{M}_{0N}^A | \mathbf{M}_{11}^B \rangle & \cdots & \langle \mathbf{M}_{0N}^A | \mathbf{M}_{1N}^B \rangle \end{pmatrix}$$
$$\times \begin{pmatrix} \langle \mathbf{M}_{11}^B | \mathbf{M}_{11}^B \rangle & \cdots & \langle \mathbf{M}_{11}^B | \mathbf{M}_{1N}^B \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{M}_{1N}^B | \mathbf{M}_{11}^B \rangle & \cdots & \langle \mathbf{M}_{1N}^B | \mathbf{M}_{1N}^B \rangle \end{pmatrix}^{-1}. \tag{21}$$

Because even modes (modes antisymmetric in z) of base A give rise to only even modes of base B, and in like manner odd modes (modes symmetric in z) only odd modes, every second element of the matrix is identically zero. The largest coefficients are typically those for $n' = n$, and $n' = n \pm 2$. To illustrate with an example, a prolate spheroid whose spheroidal parameter $kf = 0.5$ and surface $\xi_0 = 1.1547$ (corresponding to an ellipticity, or axial ratio, of $\sqrt{1-\xi_0^{-2}} \approx 0.50$), the coupling matrix including modes from $n, n' = 1, \ldots, 5$, is

$$\overline{\overline{\alpha}} \approx \begin{pmatrix} 0.990\,10 & 0 & 0.006\,60 & 0 & -0.000\,19 \\ 0 & 1.014\,19 & 0 & 0.003\,09 & 0 \\ 0.004\,23 & 0 & 0.994\,98 & 0 & 0.001\,76 \\ 0 & 0.002\,28 & 0 & 1.003\,41 & 0 \\ 0.000\,00 & 0 & 0.001\,40 & 0 & 0.997\,94 \end{pmatrix}.$$

Here, the diagonal character due to the smallness of $kf\xi_0$ is quite pronounced. In fact, in the spherical limit case $f \to 0$ the elements of the coupling matrix $\alpha_{nn'} \to \delta_{nn'}$, so that the matrix approaches the $N \times N$-identity and expansion (20) reduces to only one term, expressing essentially the relationship [10, equation (8.752.1)]

$$-\sqrt{1-\eta^2}\frac{\mathrm{d}}{\mathrm{d}\eta}P_n(\eta) = P_n^1(\eta) \qquad \text{or} \qquad \frac{\mathrm{d}}{\mathrm{d}\theta}P_n(\cos\theta) = P_n^1(\cos\theta) \tag{22}$$

as a function of the angle θ of the spherical co-ordinates.

Taking another example, where $kf = 2$ and $\xi_0 = 1.005\,04$, corresponding to a prolate of ellipticity ≈ 0.10, we have

$$\overline{\overline{\alpha}} \approx \begin{pmatrix} 0.864\,01 & 0 & 0.089\,31 & 0 & 0.001\,08 \\ 0 & 1.225\,82 & 0 & 0.057\,24 & 0 \\ 0.054\,42 & 0 & 0.929\,59 & 0 & 0.027\,11 \\ 0 & 0.039\,50 & 0 & 1.052\,94 & 0 \\ 0.000\,72 & 0 & 0.020\,97 & 0 & 0.967\,96 \end{pmatrix}.$$

The diagonal character is less apparent here than in the first example, which can be expected in consequence of the larger size-parameter $kf\xi_0$.

An indisputable disadvantage of the present conversion routine is the fact that, for a simple transformation of N modes from one base to another, one has to compute, in principle, two matrices, each containing at least N^2 elements. Moreover, to suppress the rounding errors innate in the present formulation (especially afflicting the remotest cross-coupling terms, farthest from the diagonal) a high degree of numerical accuracy is required. Fortunately the process can be facilitated by the very accurate and fast methods to compute the angular integrals given in section 2.1. The remaining angular integrals arising from the inner products of the transformation matrix will be evaluated below.

3.1. Angular integrals involved in $\langle \mathbf{M}_{0n}^A | \mathbf{M}_{1n'}^B \rangle$

To compute

$$\begin{aligned} I_4(n,n') &= \int_{-1}^{1} \eta\sqrt{1-\eta^2} S_{0n}(\eta) S_{1n'}(\eta)\,\mathrm{d}\eta \\ &= \sum_{r=0}^{\infty} d_r^{1n} \sum_{r'=0}^{\infty} d_{r'}^{1n'} \int_{-1}^{1} \eta\sqrt{1-\eta^2} P_r(\eta) P_{r'+1}^1(\eta)\,\mathrm{d}\eta \end{aligned} \tag{23}$$

benefit may be drawn from the fact that

$$\int_{-1}^{1} x\sqrt{1-x^2} P_r(x) P_{r'+1}^1(x)\,\mathrm{d}x = \begin{cases} \dfrac{-2r(r+1)}{(2r-1)(2r+1)(2r+3)} & r = r'+1 \\ \dfrac{-2(r+1)(r+2)(r+3)}{(2r+1)(2r+3)(2r+5)} & r = r'-1 \\ \dfrac{2(r-2)(r-1)r}{(2r-3)(2r-1)(2r+1)} & r = r'+3 \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

which property is proved in the appendix. Hence,

$$\begin{aligned} I_4(n,n') = 2\sum_{r=0}^{\infty} \frac{d_r^{0n}}{2r+1} \Bigg[& \frac{-r(r+1)}{(2r-1)(2r+3)} d_{r-1}^{1n'} \\ & - \frac{(r+1)(r+2)(r+3)}{(2r+3)(2r+5)} d_{r+1}^{1n'} + \frac{(r-2)(r-1)r}{(2r-3)(2r-1)} d_{r-3}^{1n'} \Bigg]. \end{aligned} \tag{25}$$

Similarly, in evaluating

$$I_5(n,n') = \int_{-1}^{1} \sqrt{1-\eta^2}\, \frac{\mathrm{d}S_{0n}(\eta)}{\mathrm{d}\eta} S_{1n'}(\eta)\,\mathrm{d}\eta \tag{26}$$

the relationship

$$\int_{-1}^{1} \sqrt{1-x^2}\, \frac{\mathrm{d}P_r(x)}{\mathrm{d}x} P^1_{r'+1}(x)\,\mathrm{d}x = -\int_{-1}^{1} P^1_r(x) P^1_{r'+1}(x)\,\mathrm{d}x = -\frac{2r(r+1)}{2r+1}\delta_{r,r'+1} \tag{27}$$

when applied repeatedly, induces

$$I_5(n,n') = -2\sum_{r=1}^{\infty} \frac{r(r+1)}{2r+1} d_r^{0n} d_{r-1}^{1n'}. \tag{28}$$

4. Conclusion

A transformation matrix has been presented, by which the vector wavefunctions of two alternative bases of rotationally symmetric spheroidal modes may be converted into each other. The rationale of the work has been to avoid complicated numerical integration schemes called upon in the treatment of some inseparable double integrals. Although the conversion process in itself involves double integrals, these are now separable into angular and radial parts. To evaluate the angular integrals, simple sum rules were presented, resulting from repeated application of the orthogonality properties of Legendre functions.

Appendix

Use of the identities (22) and (8.914.2) of [10] permits (15) to be written

$$\int_{-1}^{1} \frac{\mathrm{d}P_r(x)}{\mathrm{d}x} \frac{\mathrm{d}P_{r'}(x)}{\mathrm{d}x} (1-x^2)x^2\,\mathrm{d}x \tag{29}$$

$$= \frac{rr'(r+1)(r'+1)}{(2r+1)(2r'+1)} \int_{-1}^{1} (P_{r+1}(x) - P_{r-1}(x))(P_{r'+1}(x) - P_{r'-1}(x)) \frac{x^2\mathrm{d}x}{(1-x^2)} \tag{30}$$

which, by regrouping terms can be stated as the sum of two terms,

$$I_a(r,r') = -\frac{rr'(r+1)(r'+1)}{(2r+1)(2r'+1)} \int_{-1}^{1} (P_{r+1}(x) - P_{r-1}(x))(P_{r'+1}(x) - P_{r'-1}(x))\,\mathrm{d}x \tag{31}$$

$$I_b(r,r') = \frac{rr'(r+1)(r'+1)}{(2r+1)(2r'+1)} \int_{-1}^{1} (P_{r+1}(x) - P_{r-1}(x))(P_{r'+1}(x) - P_{r'-1}(x)) \frac{\mathrm{d}x}{(1-x^2)}. \tag{32}$$

The integral $I_a(r,r')$ becomes

$$I_a(r,r') = -\frac{2rr'(r+1)(r'+1)}{(2r+1)(2r'+1)} \left[\delta_{r,r'}\left(\frac{1}{2r+3} + \frac{1}{2r-1}\right) - \frac{\delta_{r,r'-2}}{2r+3} - \frac{\delta_{r,r'+2}}{2r-1}\right]$$

upon application of the orthogonality of the Legendre polynomials [10, equation (7.221.1)], while the integral $I_b(r,r')$ can be rewritten as

$$I_b(r,r') = \int_{-1}^{1} P^1_r(x) P^1_{r'}(x)\,\mathrm{d}x = \frac{2r(r+1)}{2r+1}\delta_{r,r'} \tag{33}$$

by using (22) and (8.914.2) of [10] and by exploiting the orthogonality of the Legendre functions [10, equation (7.112.1)]. Hence, the result (15) follows in a straightforward manner.

Next, the key expression in the integrand of (18) can developed as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{1-x^2}P_r^1(x)\right] = \sqrt{1-x^2}\frac{\mathrm{d}P_r^1(x)}{\mathrm{d}x} - \frac{xP_r^1(x)}{\sqrt{1-x^2}}$$

$$= -P_r^2(x) - \frac{2xP_r^1(x)}{\sqrt{1-x^2}} = r(r+1)P_r(x) \qquad (34)$$

using [10, equations (8.733.1) and (8.733.3)]. Expression (19) for $I_3(n, n')$ is, in analogy, obtainable using orthogonality.

Finally, to prove the relationship (24), we cast the integral in the form

$$I_c(r, r'+1) = \frac{(r'+1)(r'+2)}{(2r'+3)(2r+1)}\int_{-1}^{1}((r+1)P_{r+1}(x) + rP_{r-1}(x))(P_{r'+2}(x) - P_{r'}(x))\,\mathrm{d}x \qquad (35)$$

on account of (22) and the formulae [10, equations (8.914.1) and (8.914.2)]. Recurrent application of the orthogonality of Legendre functions [10, equation (7.221.1)] yields

$$I_c(r, r'+1) = \frac{2(r'+1)(r'+2)}{(2r'+3)(2r+1)}\left[\delta_{r,r'+1}\left(\frac{r+1}{2r+3} - \frac{r}{2r-1}\right) - \frac{(r+1)\delta_{r,r'-1}}{2r+3} + \frac{r\delta_{r,r'+3}}{2r-1}\right] \qquad (36)$$

from which the requested result (24) immediately follows.

References

[1] Marengo E A and Devaney A J 1999 *IEEE Trans. Antennas Propag.* **47** 410

[2] Flammer C 1957 *Spheroidal Wavefunctions* (Stanford, CA: Stanford University Press)

[3] Li L-W, Kang X-K and Leong M-S 2002 *Spheroidal Wavefunctions in Electromagnetic Theory* (New York: Wiley)

[4] Asano S and Yamamoto G 1975 *Appl. Opt.* **14** 29

[5] Kokkorakis G C and Roumeliotis J A 1998 *J. Electromagn. Waves Appl.* **12** 1601

[6] Schelkunoff S A 1952 *Advanced Antenna Theory* (London: Chapman and Hall)

[7] Li L-W, Leong M-S, Yeo T-S, Kooi P-S and Tan K-Y 1998 *Phys. Rev.* E **58** 6792

[8] Spence R D and Wells C P 1951 *Commun. Pure Appl. Math.* **4** 95

[9] Wall D J N 1978 *J. Phys. A: Math. Gen.* **11** 749

[10] Gradshteyn I S and Ryzhik I M 1994 *Table of Integrals, Series and Products* 5th edn (London: Academic)